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ASYMPTOTIC STABILITY AND SMOOTH EQUIVALENCE OF ORDINARY EQUATIONS

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Under certain specified conditions the asymptotic stability is a coarse property [1], (i.e. addition of fairly smooth functions to the right-hand sides of equations does not disturb the asymptotic stability). It is shown below that in this case the unperturbed system is coarse in a more general sense, namely, any smooth system acted upon by fairly small smooth perturbations, can be returned to its unperturbed state by a smooth reversible transformation. The value and order of the perturbations and the domain of existence of the transformation are all estimated explicitly. The condition required for the above assertion to hold, is that of the existence of a Liapunov function admitting, together with its derivative, specified estimates. This requirement holds, in particular, in the case when the right-hand sides of the unperturbed system are homogeneous functions, the

position of equilibrium is asymptotically stable, and its neighborhood contains no solutions bounded when $-\infty < t < \infty$ (see [1]). If the system is analytic, the requirement will hold in at least all critical cases investigated in which the asymptotic stability with $t \to \infty$ or $t \to -\infty$ is fixed, since in these cases the Liapunov function will be analytic, or simply polynomial. It follows therefore from the theorem which we prove, that in all the cases in question, the system is reduced by a smooth transformation, to the polynomial form. If the unperturbed system is linear, then from the theorem proved follows a theorem on linearization appearing in [2]; if the system is nonlinear but of second order, a theorem from [3] ensues. The results obtained in this paper for the nonlinear autonomous systems are extended to the case when the perturbations are continuous and bounded functions of time. This makes possible the investigation of the dynamics of the process in the neighborhood of asymptotically stable equilibria and of periodic modes, ignoring a wide range of external perturbations.

1. Formulation of the problem and the result. Let us consider a real autonomous system of equations with smooth right-hand sides

$$x_i^{\cdot} = f_i(x) + \varepsilon R_i(x), \quad i = 1, \ldots, n, \ |\varepsilon| < \varepsilon^*, \qquad f_i(0) = R_i(0) = 0 \tag{1.1}$$

We shall try to clarify the conditions of existence of a smooth reversible transformation near the singularity $x = 0$, which transforms the system (1.1) to the form

$$x_i^{\cdot} = f_i(x) \tag{1.2}$$

We shall also try to find the value and order of the "perturbations" $\varepsilon R_i(x)$ and to estimate the domain of existence of such a transformation.

We assume that the right-hand sides of the system (1.1) are twice continuously differentiable and satisfy the inequalities

$$|f_i| \leqslant c_1\rho^m, \quad |R_i| \leqslant c_2\rho^{m+\sigma}, \quad \left|\frac{\partial f_i}{\partial x_k}\right| \leqslant c_1\rho^{m-1}$$
$$\left|\frac{\partial R_i}{\partial x_k}\right| \leqslant c_2\rho^{m+\sigma-1}, \quad \rho = \|x\|_2^2 = (x_1^2 + \ldots + x_n^2)^{1/2} \leqslant \delta \tag{1.3}$$

We also assume that the system (1.2) admits a Liapunov function $V(x)$ which satisfies in the region $\Gamma : V(x) \leqslant d$ the inequalities

$$c_0\rho^A \leqslant V \leqslant \nu c_0\rho^A, \quad D_f V \geqslant \mu\rho^{A+m-1} \tag{1.4}$$
$$\left|\frac{\partial V}{\partial x_k}\right| \leqslant \nu c_0 \rho^{A-1}$$
$$d = \nu c_0\delta^A, \ c_0 > 0, \ \mu > 0, \ A > 0, \ \nu \geqslant 1$$

where

$$D_f = \frac{\partial}{\partial t} + \sum_k f_k \frac{\partial}{\partial x_k}$$

and D_f is the differential operator acting along the trajectory of (1.2).

We introduce the notation

$$k_0 = \frac{m-1}{A}, \quad \sigma_0 = \frac{\sigma}{A}, \quad \alpha = k_0 + \sigma_0 + \frac{1}{A}, \quad \mu_0 = \mu(\nu c_0)^{-1-k_0} \tag{1.5}$$

$$c_1^* = \frac{c_1\sqrt{n}}{c_0^{k_0}}, \quad c_2^* = \frac{c_2\sqrt{n}}{c_0^{k_0+\sigma_0}}d^{\sigma_0}, \quad C = c_2\sqrt{n}c_0^{-\alpha}$$

$$a = \frac{C'(\nu c_0)^{1/A}d^{\sigma_0+k_0-\beta}\sqrt{n}(\alpha\mu_0 - c_1^*)}{(c_2^* + \alpha\nu n c_0 c_2 d^{\sigma_0})(l-1) + C'd^{\sigma_0}c_0^{1/A}l\sqrt{n}\nu^{1/A}} \tag{1.6}$$

The parameters C', l and β are restricted by the inequalities

$$l > 1, \quad \beta \geqslant k_0, \quad C' > C \tag{1.7}$$

Theorem. If the conditions (1.3) and (1.4) hold, $|\varepsilon| < \varepsilon^*$ and $\sigma_0 > \sigma^*$, then a reversible, continuously differentiable transformation $x' = x'(x, \varepsilon)$ exists in the open region

$$\Gamma_1 : |x_i| < b \tag{1.8}$$

which transforms the system (1.1) into the system (1.2).

When $m > 1$, we have

$$\sigma^* = \max\left(1, \frac{c_1^*}{\mu_0} + l\beta - k_0 - \frac{1}{A}\right) \tag{1.9}$$

$$\varepsilon^* = \min\left(\frac{\mu_0(\alpha - l\beta) - c_1^*}{B_1}, \frac{\alpha\mu_0 - c_1^*}{B + C'(\nu c_0)^{1/A}d^{\sigma_0}\sqrt{n}/l-1}\right)$$

$$b = \frac{1}{\sqrt{n}}\left(\frac{d}{\nu c_0}\right)^{1/A} - |\varepsilon|\frac{C'd^{\alpha-\beta}}{a(l-1)}$$

and for $m = 1$ we have

$$\sigma^* = \max\left(1, \frac{c_1^*}{\mu_0} - \frac{1}{A}\right) \tag{1.10}$$

$$\varepsilon^* = \min\left(\frac{\mu_0}{\nu n c_0 c_2 d^{\sigma_0}}, \frac{\alpha\mu_0 - c_1^*}{B + C'd^{\alpha}/b_0}\right), \quad b_0 = \frac{1}{\sqrt{n}}\left(\frac{d}{\nu c_0}\right)^{1/A}$$

$$b = \left(\frac{1}{\nu c_0}\right)^{1/A}\frac{1}{\sqrt{n}} - |\varepsilon|\frac{c'd^{\alpha}}{|\lambda|}$$

where

$$B = c_2^* + \nu n c_0 c_2 d^{\sigma_0}, \quad B_1 = c_2^* + (\alpha - l\beta)\nu n c_0 c_2 d\sigma^{\sigma_0} \tag{1.11}$$

$$\lambda < 0, \quad |\lambda| < \frac{c'(\alpha\mu_0 - c_1^*)d^{\alpha}}{b_0 B + c'd^{\alpha}}$$

If $m = 1$, the functions $f_i(x)$ are linear and the eigenvalues of the linear part satisfy the conditions $0 < \mathrm{Re}\lambda_1 \leqslant \ldots \leqslant \mathrm{Re}\,\lambda_n$, then

$$\sigma^* = 0 \quad \text{for } R^i \in C^1, \quad \sigma^* = 1 \quad \text{for } R_i \in C^2 \tag{1.12}$$

$$\varepsilon^* = \frac{\mathrm{Re}\,\lambda_1}{c_2\delta^{\sigma}(\alpha n + \sqrt{n})}$$

$$b = \frac{\delta}{\sqrt{n}} - |\varepsilon|\frac{c'\delta^{1+\sigma}}{|\lambda|}$$

where

$$(1+\sigma)\,\mathrm{Re}\,\lambda_n - \mathrm{Re}\,\lambda_1 + |\varepsilon|c_2\sqrt{n}(1 + \alpha\sqrt{n}\delta)^{\sigma} < \lambda < (1+\sigma)\,\mathrm{Re}\,\lambda_n \tag{1.13}$$

The proof of this theorem which is given below (Sect. 2), utilizes the lemmas given in Sect. 3 of this paper.

2. Proof of the theorem. Let

$$x = x(x', \varepsilon', \tau), \quad \varepsilon = \varepsilon' + \tau \; (x = x(x', \varepsilon, 0) = x') \tag{2.1}$$

be a one-parameter group of transformations preserving Eq. (1.1), the point $x = 0$ and translating the parameter ε. Let also

$$\frac{\partial}{\partial \varepsilon} + X \equiv \frac{\partial}{\partial \varepsilon} + \sum_i \xi_i(x, \varepsilon) \frac{\partial}{\partial x_i}$$

the operator of algebra corresponding to this group and

$$D = \frac{\partial}{\partial t} + \sum_i (f_i + \varepsilon R_i) \frac{\partial}{\partial x_i}$$

be the operator of differentiation along the trajectory of the system (1.1). Then

$$[D, X] = R^*, \quad R^* = \sum_i R_i \frac{\partial}{\partial x_i} \tag{2.2}$$

The above operator equation has the following n corresponding scalar equations

$$\sum_j (f_j + \varepsilon R_j) \frac{\partial \xi_i}{\partial x_j} = \sum_j \left(\frac{\partial f_i}{\partial x_j} + \varepsilon \frac{\partial R_i}{\partial x_j} \right) \xi_j(x, \varepsilon) + R_j \tag{2.3}$$

Our aim is to prove the existence of a continuously differentiable solution of Eq. (2.3), and to obtain the necessary estimates.

If $\xi_i(x, \varepsilon)$ is a certain continuous solution of the system (2.3), then the final equations of the transformations (2.1) of the group will be obtained, in accordance with the Lie theorem [4], in the form of the solution $x_i = x_i(x', \varepsilon', \tau)$ of the following Cauchy problem:

$$\frac{\partial x_i}{d\tau} = \xi_i(x, \varepsilon), \quad \frac{d\varepsilon}{d\tau} = 1; \quad x_i|_{\tau=0} = x_i', \quad \varepsilon|_{\tau=0} = \varepsilon' \tag{2.4}$$

The theorems of existence and continuous dependence of the solutions on the initial parameters enable us to estimate the region in which the solution $x_i = x_i(x', \varepsilon', \tau)$ exists and is continuous in x' for $0 \leqslant \tau \leqslant \varepsilon$. By definition the transformations (2.1), $(x_i = x_i(x', \varepsilon', \tau))$ transform the system (1.1) into the system $\dot{x}_i' = f_i(x') + \varepsilon' R_i(x')$. Since $\varepsilon = \varepsilon' + \tau$, $\varepsilon' = 0$ when $\tau = \varepsilon$. Consequently the transformation $x_i = x_i(x', 0, \varepsilon)$ transforms the system (1.1) into the system $\dot{x}_i' = f_i(x')$, i.e. into (1.2).

Next we should prove the existence of a continuous solution of the system (2.3). Consider, in the space (x, t), a semi-infinite cylinder $Z = \Gamma \times t$: $V \leqslant l$, $t \geqslant 0$. If the condition (2.6) given below holds, then all trajectories of the system (1.1) except one $(x = 0)$ which penetrate into Z through its base Γ leave it through its lateral surface. Thus a segment S of the space (x, t) bounded by the set of trajectories of the system (1.1) emerging from the points $V(x) = l$ and by the plane $t = 0$, contains the whole of Z.

To obtain the condition (2.6), we find an estimate from below for the function $DV(x)$

$$DV = D_f V + \varepsilon \sum_i R_i \frac{\partial V}{\partial x_i} \geqslant \mu_0 V^{1+k_0} - |\varepsilon| \sqrt{R_1^2 + \ldots + R_n^2} \times \tag{2.5}$$

$$\sqrt{\left(\frac{\partial V}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial V}{\partial x_n}\right)^2} \geqslant \mu_0 V^{1+k_0} - |\varepsilon| n c_2 \rho^{m+\sigma} \nu c_0 \rho^{A-1} \geqslant$$

$$(\mu_0 - |\varepsilon| \nu n c_0 c_2 d^{\sigma_0}) V^{1+k_0}.$$

We obtain $DV > 0$ if

$$|\varepsilon| < \frac{\mu_0}{\nu n c_0 c_2 d^{\sigma_0}} \tag{2.6}$$

Let us consider the Cauchy problem for the following hyperbolic system:

$$D\psi_i(t, x, \varepsilon) = \sum_k \left(\frac{\partial f_i}{\partial x_k} + \varepsilon \frac{\partial R_i}{\partial x_k} \right) \psi_k(t, x, \varepsilon), \quad \psi_i(0, x, \varepsilon) = R_i(x) \tag{2.7}$$

Since $S \supset Z$ and the coefficients of $\psi_i(t, x, \varepsilon)$ appearing in the right-hand sides of the system (2.7) are continuously differentiable, then by virtue of Lemma 1 a solution $\psi_i(t, x, \varepsilon)$ of the Cauchy problem exists, is unique and continuously differentiable in Z.

By Lemma 2, the uniform convergence of the integrals

$$I_i(x, \varepsilon) = \int_0^\infty \psi_i(t, x, \varepsilon)\, dt, \quad J_{ij}(x, \varepsilon) = \int_0^\infty \frac{\partial \psi_i}{\partial x_j}\, dt$$

for $V \leqslant l$ guarantees the existence of continuously differentiable solution

$$\xi_i(x, \varepsilon) = I_i(x, \varepsilon) \tag{2.8}$$

of the system (2.3).

To prove that the integrals converge uniformly, we construct an estimate for the functions $\psi_i(t, x, \varepsilon)$. The following differential inequality holds:

$$Dv \leqslant (c_1^* + |\varepsilon| c_2^*)\, V^{k_0} v, \quad v = (\psi_1^2 + \ldots + \psi_n^2)^{1/2} \tag{2.9}$$

In fact, multiplying the i-th equation of (2.7) by ψ_i and adding all equations, we obtain

$$vDv \leqslant \sum_{i,k} \left(\left| \frac{\partial f_i}{\partial x_k} \right| + |\varepsilon| \left| \frac{\partial R_i}{\partial x_k} \right| \right) |\psi_i| |\psi_k| \leqslant$$

$$(c_1 \rho^{m-1} + |\varepsilon| c_2 \rho^{m-1+\sigma}) \sum_{i,k} |\psi_i| |\psi_k| \leqslant (c_1 \rho^{m-1} + |\varepsilon| c_2 \rho^{m-1+\sigma}) \sqrt{n} v^2$$

The first inequality of (1.4) and the definitions (1.5) now yield together the inequality (2.9).

Consider the case of $m > 1$. The function

$$u = C' V^\alpha (1 + a V^\beta t)^{-l} \tag{2.10}$$

majorizes the function $v(t, x)$ if $C' > C$ and the parameters α, β, l, C satisfy the conditions of the theorem.

We prove this assertion by assuming the opposite. We have

$$u(0, x) - v(0, x) = C' V^\alpha - \sqrt{R_1^2 + \ldots + R_n^2} > CV^\alpha -$$

$$c_2 \sqrt{n}\, \rho^{m+\sigma} \geqslant c_2 \sqrt{n}\, c_0^{-\alpha} V^\alpha - c_2 \sqrt{n}\, V^{(m+\sigma)/A} c_0^{-(m+\sigma)/A} \equiv 0$$

i.e. when $t = 0$, $v < u$. Now we assume that the difference $u - v$ becomes negative somewhere in Z. Then by virtue of the property of the trajectories of the system (1.1) guaranteed by the condition (2.6), we can find a trajectory emerging from Γ such, that we shall have $u - v > 0$ along this trajectory up to a certain point A at which we shall have $u = v$. Consequently, at the point A of this trajectory we obtain

$$u = v, \quad Dv \geqslant Du \tag{2.11}$$

Let us find the difference

$$Dv - (c_1^* + |\varepsilon| c_2^*) V^{k_0} v \geqslant Du - (c_1^* + |\varepsilon| c_2^*) V^{k_0} u = \tag{2.12}$$

$$\frac{u}{V(1+aV^{\beta}t)}\left[\alpha DV - alV^{\beta-1} + a(\alpha - l\beta)V^{\beta}tDV\right] -$$

$$(c_1^* + |\varepsilon| c_2^*) V^{k_0} u \equiv \frac{u}{V(1+aV^{\beta}t)}(\omega_1 + aV^{\beta}t\omega_2)$$

$$\omega_1 \equiv \alpha DV - alV^{(\beta+1)} - (c_1^* + |\varepsilon| c_2^*) V^{1+k_0}$$

$$\omega_2 \equiv (\alpha - l\beta) DV - (c_1^* + |\varepsilon| c_2^*) V^{1+k_0}$$

at the point A. Using (2.5) and the condition that $\beta \geqslant k_0$, we obtain

$$\omega_1 > [\alpha(\mu_0 - |\varepsilon| \nu n c_0 c_2 d^{\sigma_0}) - (c_1^* + |\varepsilon| c_2^* + ald^{\beta-k_0})] V^{1+k_0}$$

$$\omega_2 \geqslant [(\alpha - l\beta)(\mu_0 - |\varepsilon| \nu n c_0 c_2 d^{\sigma_0}) - (c_1^* + |\varepsilon| c_2^*)] V^{1+k_0}$$

By virtue of the choice of ε^* (1.9) we find that $\omega_2 \geqslant 0$, while a and ε from (1.6) yield $\omega_1 > 0$. Taking into account (2.11), we find that at the point $A \in Z$

$$Dv \geqslant Du > (c_1^* + |\varepsilon| c_2^*) V^{k_0} v$$

which contradicts the inequality (2.9). We therefore have

$$v(t, x) \leqslant u(t, x) \qquad (2.13)$$

everywhere in Z.

Let us now find an estimate for the functions $\xi_i(x, \varepsilon)$

$$\xi_i(x, \varepsilon) = I_i(x, \varepsilon) \leqslant \int_0^{\infty} u(t, x)\,dt = \frac{C'V^{\alpha-\beta}}{a(l-1)}\left[\frac{1}{(1+aV^{\beta}t)^{l-1}}\bigg|_{t=0} - \frac{1}{(1+aV^{\beta}t)^{l-1}}\bigg|_{t=\infty}\right] = \frac{C'V^{\alpha-\beta}}{a(l-1)} \qquad (2.14)$$

The definition of α and the first condition of (1.9) together yield $\alpha - \beta > 0$. Therefore the functions $\xi_i(x, \varepsilon)$ are continuous in Γ

$$|\xi_i(x, \varepsilon)| \leqslant \frac{C'V^{\alpha-\beta}}{a(l-1)} \leqslant \frac{C'd^{\alpha-\beta}}{a(l-1)} \equiv M = \max_{\Gamma} |\xi_i| \qquad (2.15)$$

The uniform convergence of the integrals $J_{ij}(x, \varepsilon)$ and, consequently, the continuous differentiability of the functions ξ_i, are proved in the similar manner.

The equations and the boundary conditions of the corresponding Cauchy problem are obtained from (2.7) by a single differentiation with respect to the variables x_j, and the estimates are derived with the help of a majorant of the form (2.10).

Let us now consider Eqs (2.4). The functions ξ_i are clearly continuously differentiable in a close cube

$$|x| \leqslant \frac{1}{\sqrt{n}}\left(\frac{d}{\nu c_0}\right)^{1/A} \equiv b_0$$

By virtue of the theorem of existence, uniqueness and differentiability with respect to the initial values, the solution $x = x(x', \varepsilon', \tau)$ of the system (2.4) is continuously differentiable with respect to x_i' and τ for $|\tau| \leqslant |\varepsilon|$ and $|x'| \leqslant b < b_0$, provided that the following condition holds:

$$|\varepsilon| < \frac{b_0 - b}{M}, \quad M = \max|\xi_i|$$

Taking into account the definition (1.9) of b and the estimate in (2.16) we see that the above condition holds since $|\varepsilon| < \varepsilon^*$. The positiveness of ε^* and a in (1.9) and (1.6) is guaranteed by the condition that $\sigma_0 > \sigma^*$, and the positiveness of b by the condition $|\varepsilon| < \varepsilon^*$. From (1.9) we see that the condition (2.6) also holds.

Let us now consider the case $m = 1$. The only difference from the previous case is in the choice of the majorizing function

$$u = C'V^\alpha e^{\lambda t}, \quad C' > C$$

At the point A where the inequality $u - v > 0$ is assumed to fail, we obtain, by virtue of the conditions (1.10),

$$Dv - (c_1^* + |\varepsilon| c_2^*) v \geqslant Du - (c_1^* + |\varepsilon| c_2^*) u =$$
$$[\lambda + \alpha (\mu_0 - |\varepsilon| \nu n c_0 c_2 d^{\sigma_0}) - (c_1^* + |\varepsilon| c_2^*)] Vu > 0$$

We can also easily establish that the quantity b defined by the last formula of (1.12) is positive owing to the condition $|\varepsilon| < \varepsilon^*$. In the present case the value $M = \max|\xi_i|$ is obtained from the estimate

$$|\xi_i| \leqslant \int_0^\infty u(t,x)\,dt = C'V^\alpha \int_0^\infty e^{\lambda t} = \frac{C'V^\alpha}{|\lambda|} < \frac{C'd^\alpha}{|\lambda|} = M$$

Assume now that $m = 1$, the functions $f_i(x)$ are linear, and the eigenvalues of the linear part of the system (1.1) satisfy the conditions

$$0 < \operatorname{Re}\lambda_1 \leqslant \ldots \leqslant \operatorname{Re}\lambda_n$$

In this particular case the estimates can be somewhat simplified and the estimate for σ improved. The system (1.1) (real roots are assumed simple) assumes, for $t_1 = -t > 0$, the form

$$dx_i/dt_1 = -\operatorname{Re}\lambda_i x_i + \operatorname{Im}\lambda_i y_i - \varepsilon R_i, \quad |R_i| \leqslant c_2\rho^{1+\sigma} \tag{2.16}$$
$$dy_i/dt_1 = -\operatorname{Im}\lambda_i x_i - \operatorname{Re}\lambda_i y_i - \varepsilon R_i, \quad i = 1, \ldots, n_1$$
$$dz_j/dt_1 = -\lambda_j z_j - \varepsilon R_j, \quad j = n_1 + 1, \ldots, n$$

The equations of the corresponding Cauchy problem in the region $\rho \leqslant \delta,\ t_1 \geqslant 0$ yield the estimate

$$D_{t_1} v \leqslant (-\operatorname{Re}\lambda_1 + |\varepsilon| c_2\sqrt{n}\delta^\sigma) v \tag{2.17}$$

where D_{t_1} denotes the differential operator acting along the trajectory of the system (2.16). The majorizing function is taken in the form

$$u = C'\rho^\alpha e^{\lambda t_1}, \quad C' > C$$

At the point A where the inequality $u - v > 0$ is assumed to fail we have $u = v$, and

$$D_{t_1} v \geqslant D_{t_1} u = [\lambda - \alpha(\operatorname{Re}\lambda_n + |\varepsilon| c_2 n\delta^\sigma)] v$$

If λ is chosen according to the condition (1.13) and $|\varepsilon| < \varepsilon^*$, then

$$D_{t_1} v > (-\operatorname{Re}\lambda_1 + |\varepsilon| c_2\sqrt{n}\delta^\sigma) v$$

which contradicts the inequality (2.17). Therefore in Γu definitely majorizes $v : v \leqslant u$. Returning to the variable t we obtain at $t \leqslant 0$

$$u = C'\rho^\alpha e^{-\lambda t}$$

According to Lemma 3, the solution of (2.3) is given by the formula

$$\xi_i(x) = \int_0^{T(x)} \psi_i(t,x)\,dt$$

where ψ_i is the solution of the Cauchy problem (2.7) and the function $T(x)$ represents any stationary solution of the equation $DT = 1$. Let us consider the solution of this equation with the boundary condition $T(x)|_{\rho=\delta} = 0$. Since we have the estimate

$$(\operatorname{Re}\lambda_1 - |\varepsilon| c_2 n\delta^\sigma)\rho \leqslant \frac{d\rho}{dt} \leqslant (\operatorname{Re}\lambda_n + |\varepsilon| c_2 n\delta^\sigma)\rho \qquad (2.18)$$

then the system (1.1) is asymptotically stable as $t \to -\infty$ when the condition (1.12) holds. According to Lemma 4, $T(x)$ is the time taken by the representative point of the system (1.1) with the initial conditions on the sphere $\rho = \delta$ to arrive at the point with coordinates $x_1, \ldots, x_n$. This fact alone is sufficient to prove rigorously that $T(x)$ is continuous within the sphere $\rho \leqslant \delta$ with the point $\rho = 0$ excluded. Integrating the inequality (2.18) from 0 to $T(x)$, we obtain

$$(\operatorname{Re}\lambda_1 - |\varepsilon| c_2 n\delta^\sigma) T(x) \leqslant \ln\left|\frac{\rho}{\delta}\right| \leqslant (\operatorname{Re}\lambda_n + |\varepsilon| c_2 n\delta^\sigma) T(x)$$

This shows that in the region Γ where $\rho \leqslant \delta$

$$\frac{1}{\operatorname{Re}\lambda_n + |\varepsilon| c_2 n\delta^\sigma} \ln\left|\frac{\rho}{\delta}\right| \leqslant T(x) \leqslant 0$$

The estimate for $\xi_i(x, \varepsilon)$ yields

$$|\xi_i| \leqslant \left|\int_0^{T(x)} \psi_i(t, x, \varepsilon)\, dt\right| \leqslant \int_{T(x)}^{0} u\, dt = \frac{C'\rho^\alpha}{\lambda}(e^{-\lambda T(x)} - 1)$$

The choice of λ is guaranteed, in accordance with (1.13), by the condition $|\varepsilon| < \varepsilon^*$

$$(1+\sigma)\operatorname{Re}\lambda_n - \operatorname{Re}\lambda_1 + |\varepsilon| c_2\delta^\sigma ((1+\sigma)n + \sqrt{n}) < (1+\sigma)\operatorname{Re}\lambda_n$$

Since $\lambda > 0$, we have

$$|\xi_i| \leqslant \frac{C'\rho^\alpha}{\lambda} \exp \frac{\lambda}{\operatorname{Re}\lambda_n + |\varepsilon| c_2 n\delta^\sigma} \ln\left|\frac{\delta}{\rho}\right| =$$

$$\frac{C'\rho^\alpha}{\lambda}\left(\frac{\delta}{\rho}\right)^{\lambda/(\operatorname{Re}\lambda_n + |\varepsilon| c_2 n\delta^\sigma}$$

The choice of λ according to formula (1.13) implies, that

$$\varkappa \equiv \alpha - \frac{\lambda}{\operatorname{Re}\lambda_n + |\varepsilon| c_2 n\delta^\sigma} > 0 \qquad (2.19)$$

therefore

$$|\xi_i| \leqslant \frac{C'\delta^{\alpha-\varkappa}}{\lambda}\rho^\varkappa \leqslant \frac{C'\delta^\alpha}{\lambda} \equiv M \qquad (2.20)$$

The formulas (2.19) and (2.20) show that the functions are continuous in the sphere $\rho \leqslant \delta$ including the point $\rho = 0$, for any $\sigma > 0$. The choice of λ according to (1.13) ensures that $b > 0$, since C' can be made to differ from C as little as required.

3. Proof of the lemmas. Consider the Cauchy problem (Eqs. (2.7))

$$D\psi_i(t, x, \varepsilon) = \sum_k \left(\frac{\partial f_i}{\partial x_k} + \varepsilon\frac{\partial R_i}{\partial x_k}\right)\psi_k(t, x, \varepsilon), \; \psi_i(0, x, \varepsilon) = R_i(x)$$

Lemma 1. The solution of the Cauchy problem in the region Z is well defined, unique and continuously differentiable. The idea of proving such assertions for the hyperbolic type systems was first proposed by Petrovskii [5] and is as follows. Since D is a differential operator acting along the trajectories of the system (1.1), $D \equiv d / dt$, i.e. by virtue of (1.1) D is a total derivative. Let $x = u(t, x_0)$ be a general solution of this system, and $x_0 = u(0, x_0)$. Then $x_0 = u(-t, x)$.

Consider the region S of the space (x, t) formed by the set of trajectories of (1.1) emerging from Γ $(V(x) \leqslant d)$ at $t = 0$. As we said before, $S \supset Z$. Let us fix in Z an arbitrary point (t, x) and denote by L the part of the trajectory between the point (t, x) and its intersection at some point $(0, x_0)$ with the plane $t = 0$. Integrating the equations of (2.7) along the curve L, we obtain

$$\psi_i(t, x, \varepsilon) - \psi_i(0, x_0, \varepsilon) = \int\limits_L \sum_k \left(\frac{\partial f_i}{\partial x_k} + \varepsilon \frac{\partial R_i}{\partial x_k}\right) \psi_k \, dt$$

and for every $x \in \Gamma$ we have $x_0 = u(-t, x) \in \Gamma$. By virtue of the initial conditions

$$\psi_i(t, x, \varepsilon) = R_i(u(-t, x)) + \int\limits_L \left(\sum_k \frac{\partial f_i}{\partial x_k} + \varepsilon \frac{\partial R_i}{\partial x_k}\right) \psi_k \, dt \qquad (3.1)$$

It is clear that every solution of the Cauchy problem (2.7) is also a solution of the integral equations (3.1) and vice versa. To continue with the proof, we use the following system of successive approximations:

$$\psi_1^{(0)}(t, x, \varepsilon) = R_i(u(-t, x))$$

$$\psi_i^{(j+1)}(t, x, \varepsilon) = R_i(u(-t, x)) + \int\limits_L \sum_k \left(\frac{\partial f_i}{\partial x_k} + \varepsilon \frac{\partial R_i}{\partial x_k}\right) \psi_k^{(j)} \, dt$$

and we find that the proof does not differ at all from that given in [4] for a second order system. It appears that a numerical majorizing convergent series can be constructed, and this implies the uniform convergence of the series

$$\psi_i^{(0)}(t, x) + \sum_{j=0}^{\infty} (\psi_i^{(j+1)}(t, x, \varepsilon) - \psi_i^{(j)}(t, x, \varepsilon))$$

which proves that the solution of the Cauchy problem,is continuous in Z

Lemma 2. Let $\psi_i(t, x, \varepsilon)$ be a solution of the Cauchy problem, let the integrals I_i and J_{ij} converge uniformly for $V(x) \leqslant l$ and let $\lim\limits_{t \to \infty} \psi_i(t, x) = 0$. Then the functions

$$\xi_i(x, \varepsilon) = \int\limits_0^{\infty} \psi_i(t, x, \varepsilon) \, dt$$

yield a continuously differentiable solution of the system (2.3).

Proof. Clearly it is sufficient to show that the functions $\xi_i(x, \varepsilon)$ satisfy the system (2.3). Integrating Eqs. (2.3) from 0 to ∞ and taking into account the initial conditions, we obtain

$$\sum_k \left(f_k^* \frac{\partial \xi_i}{\partial x_k} - \xi_k \frac{\partial f_i^*}{\partial x_k}\right) = \int\limits_0^{\infty} \sum_k \left(f_k^* \frac{\partial \psi_i}{\partial x_k} - \psi_k \frac{\partial f_i^*}{\partial x_k}\right) dt =$$

$$- \int\limits_0^{\infty} \frac{\partial \psi_i}{\partial t} \, dt = \psi_i(0, x, \varepsilon) - \psi_i(\infty, x, \varepsilon) = R_i(x)$$

Lemma 3. Let $\psi_i(t, x, \varepsilon)$ be a solution of the Cauchy problem and $T(x)$ be an arbitrary stationary solution of the equation $DT = 1$. Then the functions

$$\xi_i(x, \varepsilon) = \int\limits_0^{T(x)} \psi_i(t, x, \varepsilon) \, dt$$

yield a solution of the system (2.3).

Proof.

$$\sum_k \left(f_k^* \frac{\partial \xi_i}{\partial x_k} - \xi_k \frac{\partial f_i^*}{\partial x_k} \right) = \sum_k f_k^* \left(\int_0^{T(x)} \frac{\partial \psi_i}{\partial x_k} + \psi_i (T, x) \frac{\partial T}{\partial x_k} \right) -$$

$$\int_0^{T(x)} \sum_k \psi_k \frac{\partial f_i^*}{\partial x_k} dt = - \int_0^{T(x)} \frac{\partial \psi_i}{\partial t} dt + \psi_i (T(x), x, \varepsilon) DT =$$

$$\psi_i (0, x, \varepsilon) + \psi_i (T, x, \varepsilon)(DT - 1) = R_i(x)$$

Corollary. Let $\psi_i^* (t, x)$ be any solution of Eqs. (2.7) and let $T_1 (x)$ and $T_2 (x)$ be the two different solutions of the equation $DT = 1$. Then

$$\xi_i^* (x) = \int_{T_1(x)}^{T_2(x)} \psi_i^* (t, x)\, dt$$

are components of the operator

$$Y = \sum_i \xi_i^* (x) \partial / \partial x_i$$

of the one-parameter symmetry group of the system (1.1). The proof follows from combining the relations

$$[D, X_1] = R^*, \ [D, X_2] = R^*, \ Y = X_2 - X_1 : [D, Y] = 0$$

Lemma 4. Let $T (x)$ be any solution of the equation $DT = 1$. Then $t = T (x) - T (x_0)$ is the time of motion along the trajectory of the system (1.1) from the point x_0 to the point x.

Proof.

$$1 = DT = \frac{d}{dt} T (u (x_0, t))$$

From this follows $T (u (x_0 t)) = t + \Phi (x_0)$. When $t = 0$, $\Phi (x_0) = T (x_0)$, consequently $t = T (x) - T (x_0)$.

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